



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 11

Exercise 1:

Let V be a smooth vector field on a smooth manifold M , let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma: J \rightarrow M$ be an integral curve of V . Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\tilde{\gamma}: \tilde{J} \rightarrow M, t \mapsto \gamma(at)$$

is an integral curve of the vector field $\tilde{V} := aV$ on M , where $\tilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\hat{\gamma}: \hat{J} \rightarrow M, t \mapsto \gamma(t+b)$$

is also an integral curve of V on M , where $\hat{J} := \{t \in \mathbb{R} \mid t+b \in J\}$.

Exercise 2 (to be submitted by Thursday, 04.12.2024, 16:00):

(a) Compute the integral curves of the Euler vector field V on \mathbb{R}^n .

(b) Consider the smooth manifold $M \subset \mathbb{R}^2$ and the smooth vector field $Y \in \mathfrak{X}(M)$ from [Exercise Sheet 10, Exercise 2(d)]. Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

(c) Compute the flow of each of the following smooth vector fields on \mathbb{R}^2 :

(i) $U = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

(ii) $V = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$.

(iii) $W = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$.

- (d) Consider the smooth vector field $X_2 \in \mathfrak{X}(\mathbb{R}^3)$ from [*Exercise Sheet 10, Exercise 2(c)*]. Compute its flow and examine whether it is a complete vector field on \mathbb{R}^3 .

Exercise 3:

Let $\theta: \mathbb{R} \times M \rightarrow M$ be a smooth global flow on a smooth manifold M . Show that the infinitesimal generator V of θ is a smooth vector field on M , and that each curve $\theta^{(p)}: \mathbb{R} \rightarrow M$ is an integral curve of V .

Exercise 4:

- (a) *Naturality of integral curves:* Let $F: M \rightarrow N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F -related if and only if F takes integral curves of X to integral curves of Y .
- (b) *Naturality of flows:* Let $F: M \rightarrow N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y . Show that if X and Y are F -related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t :

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ \theta_t \downarrow & & \downarrow \eta_t \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

- (c) *Diffeomorphism invariance of flows:* Let $F: M \rightarrow N$ be a diffeomorphism. Show that if $X \in \mathfrak{X}(M)$ and if θ is the flow of X , then the flow of $F_*X \in \mathfrak{X}(N)$ is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Definition. Let V be a (rough) vector field on a smooth manifold M . A point $p \in M$ is called a *singular point* of V if $V_p = 0 \in T_pM$; otherwise, it is called a *regular point* of V .

Exercise 5:

Let V be a smooth vector field on a smooth manifold M and let $\theta: \mathfrak{D} \rightarrow M$ be the flow generated by V . Prove the following assertions:

- (a) If $p \in M$ is a singular point of V , then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.
- (b) If $p \in M$ is a regular point of V , then $\theta^{(p)}: \mathfrak{D}^{(p)} \rightarrow M$ is a smooth immersion.
[Hint: Argue by contraposition and use the fundamental theorem on flows.]